FIRST-ORDER INFINITESIMAL MECHANISMS

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Abstract—This paper discusses the analytical conditions under which a pin-jointed assembly, which has s independent states of self-stress and m independent mechanisms, tightens up when its mechanisms are excited. A matrix algorithm is set up to distinguish between first-order infinitesimal mechanisms (which are associated with second-order changes of bar length) and higher-order infinitesimal or finite mechanisms. It is shown that, in general, this analysis requires the computation of s quadratic forms in m variables, which can be easily computed from the states of self-stress and mechanisms of the assembly. If any linear combination of these quadratic forms is sign definite, then the mechanisms are first-order infinitesimal. An efficient and general algorithm to investigate these quadratic forms is given. The calculations required are illustrated for some simple examples.

Many assemblies of practical relevance admit a single state of self-stress ($s = 1$), and in this case the algorithm proposed is straightforward to implement.

This work is relevant to the analysis and design of pre-stressed mechanisms, such as cable systems and tensegrity frameworks.

1. INTRODUCTION

Assemblies of pin-jointed bars exhibit a wide range of mechanical phenomena. Such assemblies are generally described geometrically in terms of the numbers of bars and joints; but their mechanical performance can only be understood properly in terms of the numbers of *inextensional mechanisms*, m ($\geqslant 0$) and of *states of self-stress*, s ($\geqslant 0$). For a given assembly, the values of m and s may be determined by Linear Algebra techniques from the equilibrium matrix set up in the initial configuration.

In many practical cases the structural engineer will want to avoid assemblies with $m > 0$, since they will not be rigid; and thus if a proposed assembly turns out to have $m > 0$ it will be re-designed in order to make $m = 0$. Increasingly, however, engineers are becoming interested in pre-stressed mechanisms such as cable-nets and tensegrities; these assemblies have $m > 0$, but also $s > 0$. There are well-known examples in which the activation—by means of, e.g. a turnbuckle—of the single state of self-stress in an assembly having $s = 1$ stiffens, or stabilizes, all of the independent mechanisms $m > 0$ (Calladine, 1978).

The simplest example of such an assembly is shown in Fig. 1a. As in all of the examples in the present paper, all bars and joints are constrained to lie in a plane. The matrix–algebra
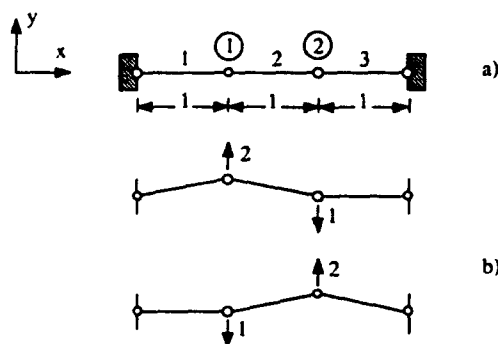

Fig. 1. (a) Planar assembly with $j = 4$ (only unconstrained nodes have numbers in the figure). $k = 4$ and $b = 3$. The vector of axial forces is $\mathbf{t} = [t_1 \ t_2 \ t_3]^T$; the vector of nodal components of displacement is $\mathbf{d} = [d_{1x}, d_{1y}, d_{2x}, d_{2y}]^T$. As shown in Section 4 of our previous paper, this assembly has $s = 1$ and $m = 2$. (b) Shows the two inextensional mechanisms and the corresponding product forces associated with them.
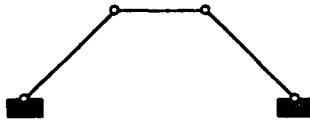
Fig. 2. Planar assembly with $j = 4$, $k = 4$ and $b = 3$. This assembly has $s = 0$ and $m = 1$; clearly its mechanism is finite.

tests described by Pellegrino and Calladine (1986) give $m = 2$, $s = 1$ for this assembly; and it may readily be seen that an in-line pretension in all three bars imparts some first-order stiffness to the two independent mechanisms. In contrast the assembly of Fig. 2 has $m = 1$, $s = 0$, and the single mechanism preserves its freedom even for large angular displacements: it is therefore a *finite* mechanism. The assembly of Fig. 1a is obviously not a finite mechanism: even if pre-stress is not activated in the initial configuration, the assembly tightens up when its mechanisms are exercised.

Many assemblies with $m > 0$, $s > 0$ tighten up after a small "inextensional" displacement. Indeed, Möbius (1837) and Maxwell (1864) knew that in general a pin-jointed assembly consisting of j joints requires at least $2j-3$ bars to make it stiff, but a lower number of bars may be sufficient if at least one bar has maximum, or minimum, length. The stiffness of such special assemblies, Maxwell warned, is "of an inferior order, ... a small disturbing force may produce a displacement infinite in comparison to itself". Mohr (1885), Föppl (1912), Kötter (1912) and subsequently Pollaczek-Geiringer (1927) were interested in the detection of these special cases. It is obviously desirable for the engineer to be able to identify a given asssembly as having a finite mechanism, as distinct from one which tightens up as its mechanisms are mobilized.

In Pellegrino and Calladine (1986, which we shall refer to henceforth as "our previous paper") we made some progress in devising algorithms which can discriminate between these various situations in a given pin-jointed assembly. Thus we showed that, given an assembly with $m > 0$ and $s > 0$, if a state of self-stress can impart positive first-order stiffness to every mechanism, then the mechanisms are *first-order infinitesimal*, i.e. they are associated with second-order changes of bar lengths. If on the other hand there are some mechanisms which cannot be stabilised by a state of self-stress, these mechanisms are *second-order infinitesimal* (at least), i.e. they are associated with third-order (or higher) length changes, or are *finite*.

Recently, in an article in the *Journal of Applied Mechanics*, Kuznetsov (1989) attacked some aspects of our work. Kuznetsov's comments have stimulated our thinking, and the present note describes some recent advances which we have made.

We shall be concerned entirely with discriminating between first-order infinitesimal mechanisms and all other types, which for the sake of compactness we shall refer to as "finite" mechanisms. We shall not consider further questions of detection of higher-order mechanisms, which have been discussed by Koiter (1984), Pellegrino (1986) and Kuznetsov (1988).

Given a pin-jointed assembly, the first stage of the matrix algorithm described in our previous paper is to compute s independent states of self-stress and m independent inextensional mechanisms. This analysis is conducted within the context of a small-deflection theory and hence all that we know about mechanisms computed in this way is that they cause no first-order changes in length of the bars. In this paper we are concerned only with assemblies for which $s > 0$ and $m > 0$.

A state of self-stress is then imposed onto the assembly, and the second stage of the algorithm is to compute, for each mechanism, the set of out-of-balance nodal forces which are required to restore equilibrium, after imparting a unit magnitude of the chosen mechanism. A total of m *product forces* are obtained in this way.

The third stage is to assemble and analyse a modified equilibrium matrix, which gives the response of the assembly to arbitrary external loads (Pellegrino and Calladine, 1984; Pellegrino, 1988). If this matrix has full rank, then the mechanisms are first-order infinitesimal; however this calculation must be supplemented by a *sign check* that the scalar
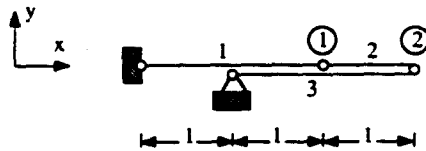
Fig. 3. Planar assembly with $j = 4$, $k = 4$ and $b = 3$; see caption of Fig. 1a for details of t, d.

product of all mechanisms with the corresponding product forces is always a positive number, which is the case if all inextensional deformations are stable. In more precise terms, the condition that the rank of the modified equilibrium matrix should be full is *necessary*, but not sufficient, for first-order infinitesimal mechanisms.

In relatively simple examples, such as those shown in Figs 1a and 3, it is easy to spot whether any mechanisms exist for which the sign check would not be satisfied. However—and this is the esssence of Kuznetsov's criticism of our previous work—any assembly with two or more independent mechanisms and also, possibly, more than one independent state of self-stress, warrants a more formal procedure.

In the present paper we address the question of computing the sign of the scalar product of every possible product force and its corresponding mechanism. This approach produces a quadratic form, which must be tested for sign definiteness. The computations are straightforward in the case of assemblies having a single state of self-stress, and indeed all of the examples previously discussed in the literature are of this kind. On the other hand, if there are several independent states of self-stress we have to deal with a linear combination of quadratic forms: we shall explain how to do this, and we shall give examples. We shall also point out in Section 5 how the present work is related to that of previous authors, and in particular to an early study by Kötter (1912).

2. SELF-STRESSING FOR POSITIVE STIFFNESS

Let us consider a planar pin-jointed assembly which consists of j *joints*, connected by a total of k *kinematic constraints* to a rigid foundation, and b *bars*. Let t be a b-dimensional vector of bar axial forces and let d be a $(2j-k)$-dimensional vector of nodal displacement components.

We compute a set of independent states of self-stress $t_1, t_2, \ldots, t_s$ following the procedure described in Section 2 of our previous paper. Clearly, any linear combination of these states of self-stress is still self-equilibrated, hence the most general self-stress state is given by:

$$t_1 \alpha_1 + t_2 \alpha_2 + \cdots + t_s \alpha_s, \tag{1}$$

where the scalar coefficients $\alpha_1, \ldots, \alpha_s$ are free to take any real value. We also compute a set of independent inextensional mechanisms for the assembly: $d_1, d_2, \ldots, d_m$. Assuming, for the sake of simplicity, that the kinematic constraints suppress all rigid-body displacements, and therefore that all vectors d_r represent internal mechanisms, we can similarly express the most general internal mechanism as:

$$d_1 \beta_1 + d_2 \beta_2 + \cdots d_m \beta_m \tag{2}$$

where the scalar coefficients $\beta_1, \ldots, \beta_m$ can again take arbitrary values. From now on, it will be convenient to write eqn (2) in the form $D\beta$, having introduced a $(2j-k) \times m$ matrix D, whose columns are the m inextensional mechanisms, and β is the column vector $[\beta_1, \ldots, \beta_m]^T$.

We begin by considering an assembly which has an arbitrary number of mechanisms $m > 0$, but only one state of self-stress $s = 1$. Let us give the assembly the state of self-stress $t = t_1$, and then impose a small inextensional displacement. The self-stressing tensions remain—to the first-order—unchanged, but they are no longer self-equilibrated because

of configuration change. Let p_{1i} be the $(2j-k)$-dimensional vector of product forces, which are required to restore nodal equilibrium of the assembly carrying the axial forces $t = t_1$, after a unit amplitude of mechanism d_i has been imposed. Detailed formulae for the components of p_{1i} can be found in our previous paper. The vector of product forces associated with the general mechanism (2) is

$$P_1\beta = p_{11}\beta_1 + p_{12}\beta_2 + \cdots + p_{1m}\beta_m. \tag{3}$$

where the $(2j-k) \times m$ matrix P_1 contains the product-force vectors arranged by columns. The subscript 1 is a reminder that the state of self-stress t_1 has been imposed. As explained in Section 1, our test for first-order infinitesimal mechanisms is that, as a result of the self-stress t, all mechanisms are endowed with positive stiffness. To ensure this, we check that the scalar product of a general inextensional mechanism $D\beta$ and the corresponding product–force vector $P_1\beta$ is *positive for all* βs :

$$\beta^T P_1^T D\beta > 0, \quad \forall \beta \in \mathcal{H}^m - 0. \tag{4}$$

Since the $m \times m$ matrix $Q_1 = P_1^T D$ is *symmetric* (this property is not immediately obvious, but can be verified by substituting expressions for all product forces given in our previous paper, and then doing the matrix multiplication), our test is equivalent to showing that the quadratic form $\beta^T Q_1 \beta$ is *positive definite*. This can be done by any of several techniques available in the literature, see e.g. Strang (1980). In Sections 3 and 4 we shall make use of the following two alternative necessary and sufficient conditions for positive definiteness of a symmetric matrix Q_1 :

(i) the pivots obtained when a Gaussian elimination is performed on Q_1 are all positive;
(ii) the eigenvalues of Q_1 are all positive.

Clearly, if Q_1 turned out according to this procedure to be *negative definite*, then a positive definite quadratic form would correspond to the self-stress $t = -t_1$; in either case the given assembly is a first-order infinitesimal mechanism. In all other cases the assembly is a "finite" mechanism.

For *assemblies with s > 1*, greater freedom is available when choosing the initial state of self-stress t. Of course, any chosen set of coefficients α_i defines, through eqn (1), a unique state of self-stress, and then we could perform the foregoing analysis for that particular t. Then, if the quadratic form Q corresponding to t is positive (or, indeed, negative) definite, our test has succeeded. It could be shown that the approaches of Kuznetsov (1975a), Besseling (1979), and Tanaka and Hangai (1986) are essentially equivalent to following this line. Clearly, if the form Q obtained in this way for a given t is not sign definite, it cannot be excluded that a different choice of α_i in eqn (1) would produce a positive definite Q : in which case the assembly is first-order infinitesimal. This difficulty highlights the need for a more general procedure which includes *all* possible states of self-stress and the corresponding quadratic forms. The remainder of this section develops the rather simple "theory" required for such a general approach.

We need to introduce the product–force vector for the general state of self-stress in eqn (1), and for the mechanism $D\beta$. Because the expressions for the product forces in our previous paper are linear in the stress terms, the product–force vector due to a linear combination of some basic states of self-stress t_i and the mechanism $D\beta$, is equal to a linear combination of the product-force vectors associated with each t_i separately. In symbols, the product–force vector corresponding to eqn (1) and to the inextensional displacement $D\beta$ can be written in the form :

$$\left(\sum_{i=1}^{s} P_i x_i\right)\beta. \tag{5}$$

Here, the $(2j-k) \times m$ matrix P_i contains the m product-force vectors for the self-stress $t = t_i$, and for the inextensional displacements $d_1, \ldots, d_m$. In analogy with eqn (4), our test for positive stiffness against a general inextensional displacement $D\beta$ becomes:

$$\beta^T \left(\sum_{i=1}^{s} P_i^T D x_i\right)\beta > 0, \quad \forall \beta \in \mathcal{R}^m - 0 \tag{6}$$

for at least one set of x_i. The above test is equivalent to checking for the existence of at least one linear combination of the matrices $Q_1 = P_1^T D, \ldots, Q_s = P_s^T D$, all symmetric and of size $m \times m$, which is positive definite. In the next section we present some examples for which the properties of the matrix:

$$Q = \sum_{i=1}^{s} Q_i x_i \tag{7}$$

can be determined easily. Then, in Section 4 we describe a general way to analyze Q.

3. EXAMPLES

In this section we shall consider several applications of the theory developed in Section 2. Figure 1a shows a three-bar assembly with $s = 1$ state of self-stress, $t_1 = [1 \ 1 \ 1]^T$ and $m = 2$ inextensional mechanisms $d_1 = [0 \ 1 \ 0 \ 0]^T$, $d_2 = [0 \ 0 \ 0 \ 1]^T$. The corresponding product-force vectors, shown in Fig. 1b, are $p_{11} = [0 \ 2 \ 0 \ -1]^T$, $p_{12} = [0 \ -1 \ 0 \ 2]^T$. All these values can be verified by inspection, but a more formal derivation can be found in our previous paper. We assemble the matrices D and P_1, which contain, respectively, the two mechanisms and the product-force vectors corresponding to those mechanisms (and to the self-stress $t = t_1$):

$$D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 0 \\ 2 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix},$$

and form the symmetric matrix:

$$Q_1 = P_1^T D = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

It is easy to verify that positive pivots are obtained when a Gaussian elimination is performed on Q_1; therefore the test (4) is satisfied. Thus, the assembly of Fig. 1 has been shown to be a first-order infinitesimal mechanism.

Figure 3 shows another assembly consisting of three collinear bars, but now the bar lengths are no longer equal and, rather more importantly, the "direction" of the last bar has been reversed. This example still has $s = 1$ state of self-stress $t_1 = [1 \ 1 \ -1]^T$; its $m = 2$ mechanisms have components identical to the first example. The matrices D and P_1 for this assembly are:
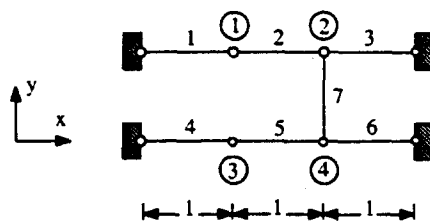
Fig. 4. Assembly consisting of two simple units, bars 1, 2, 3 and 4, 5, 6, which are identical to the assembly of Fig. 1. Bar 7 links the two units, thus suppressing one of the $2+2 = 4$ independent mechanisms resulting from Fig. 1. The eight-dimensional vector **d** and the seven-dimensional vector **t** are defined by analogy with the caption of Fig. 1a.

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} 0 & 0 \\ 3/2 & -1 \\ 0 & 0 \\ -1 & 1/2 \end{bmatrix},$$

which yield

$$\mathbf{Q}_1 = \mathbf{P}_1^T \mathbf{D} = \begin{bmatrix} 3/2 & -1 \\ -1 & 1/2 \end{bmatrix}.$$

A Gaussian elimination on the matrix $\mathbf{Q}_1$ produces the pivots $3/2$ and $-1/6$, and hence $\mathbf{Q}_1$ is *sign indefinite*. We therefore conclude that the assembly of Fig. 3 is a "finite" mechanism.

The above examples have been analyzed by Kuznetsov (1975a) by a rather different method, which involves the first- and second-order derivatives of the constraint equations enforced by each bar. It is interesting to note that, in spite of clear formal differences between the present approach and Kuznetsov's, exactly the same quadratic forms are obtained. The two procedures are in fact equivalent, although the introduction of product forces enables us to avoid the complications of the standard second-order analysis, and to implement the calculations, instead, in terms of matrices.

The next example, shown in Fig. 4, is more complicated, and has been constructed so as to have $s > 1$. It consists of two "units", each identical to the assembly of Fig. 1, and connected by a vertical bar. Clearly this assembly has $s = 2$ independent states of self-stress: $\mathbf{t}_1 = [1\ 1\ 1\ 0\ 0\ 0\ 0]^T$, $\mathbf{t}_2 = [0\ 0\ 0\ 1\ 1\ 1\ 0]^T$. The components of its $m = 3$ mechanisms can be inferred from the first example, and the resulting matrix **D** is:

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T.$$

The product-force vectors corresponding to the mechanisms in **D**, and to the states of self-stress $\mathbf{t} = \mathbf{t}_1$ and $\mathbf{t} = \mathbf{t}_2$, are respectively

$$\mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}.$$

Hence we obtain the following two symmetric matrices, corresponding to the two distinct states of self-stress:
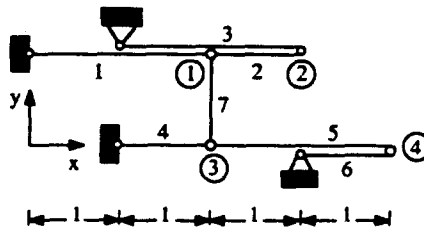
Fig. 5. Assembly consisting of two simple units linked by a bar. The first constituent unit (bars 1, 2, 3) is identical to Fig. 3; the second unit (bars 4, 5, 6), similar to Fig. 3, is made from bars of different lengths.

$$Q_1 = P_1^T D = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad Q_2 = P_2^T D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Following Section 2, we now consider, eqn (7),

$$Q = Q_1 \alpha_1 + Q_2 \alpha_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \alpha_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \alpha_2,$$

and try to find a set (α_1, α_2) for which Q is positive definite. This turns out to be a particularly easy task on this occasion: choose $\alpha_1 = \alpha_2 = 1$ (i.e. equal tensile pre-stress in each unit) and, by Gerschgorin's theorem (Strang, 1980), the eigenvalues of Q must lie in the interval (1,6), and hence must be positive. We can therefore conclude that the assembly of Fig. 4 is a first-order infinitesimal mechanism.

Lastly, we consider the assembly shown in Fig. 5, which consists of two units based on the example of Fig. 3. This assembly also has $s = 2$ independent states of self-stress $t_1 = [1 \ 1 \ -1 \ 0 \ 0 \ 0 \ 0]^T$, $t_2 = [0 \ 0 \ 0 \ 1 \ 1 \ -1 \ 0]^T$ and $m = 3$ mechanisms. Following the usual procedure we calculate the matrices Q_1 and Q_2, and consider:

$$Q = \begin{bmatrix} 3/2 & -1 & 0 \\ -1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \alpha_1 + \begin{bmatrix} 3/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & -1/2 \end{bmatrix} \alpha_2.$$

In contrast to the previous example, we have been unable to spot a set of α,s for which Q is positive definite; we have therefore performed a Gaussian elimination on the matrix

$$Q = \begin{bmatrix} \frac{3}{2}(\alpha_1 + \alpha_2) & -\alpha_1 & -\alpha_2/2 \\ -\alpha_1 & \alpha_1/2 & 0 \\ -\alpha_2/2 & 0 & -\alpha_2/2 \end{bmatrix}.$$

The first pivot is positive if $\alpha_1 + \alpha_2 > 0$. The second pivot is positive if $\alpha_1(3\alpha_2 - \alpha_1) > 0$. These two inequalities are satisfied only by the points (α_1, α_2) in the region of the plane α_1, α_2 defined by $\alpha_1 > 0$ and $\alpha_2 > \alpha_1/3$. In this region, the third pivot is positive if $4\alpha_2^2 + 3\alpha_1\alpha_2 - \alpha_1^2 < 0$; however, it is easy to show that this third inequality cannot be satisfied by any point in the region defined above. Therefore the matrix Q is *sign indefinite* for all α,s, and hence the assembly of Fig. 5 is a "finite" mechanism.

4. AUTOMATIC SEARCH FOR A POSITIVE DEFINITE Q

The last example highlights the difficulty of using Gaussian elimination to handle matrices of the type (7) ; even a small matrix of that type leads to complex sets of inequalities, which may prove difficult to solve. [But we should point out that Gaussian elimination is perfectly adequate if the linear combination (7) reduces to one matrix only, i.e. for $s = 1$, and it could also be used to analyze linear combinations of s semi-definite matrices. In this case a positive definite Q exists if and only if, the $m \times (s \cdot m)$ adjoint matrix $Q_1|Q_2| \ldots |Q_s$ spans $\mathscr{H}^m$, which can be checked by Gaussian elimination.]

In this section we describe a general algorithm which, after a sequence of operations, either identifies a positive definite Q or else shows that no such combination exists.

The algorithm is based on the idea that, if a positive definite $Q = \sum_{i=1}^{s} Q_i x_i$ exists, it will be possible to find at least one set of x_is which satisfy the n inequalities:

$$\beta_j^T \left(\sum_{i=1}^{s} Q_i x_i \right) \beta_j > 0, \quad j = 1, \ldots, n; \tag{8}$$

for any given non-vanishing vectors $\beta_j \in \mathscr{H}^m$. [Initially, $n = m$ and the vectors β_j coincide with the standard basis of $\mathscr{H}^m$ (Strang, 1980).] Once a set of x_is has been found, we calculate the eigenvalues and eigenvectors of the matrix Q thus identified. The following three cases can occur: (i) *all eigenvalues are positive*, and hence Q is positive definite; or (ii) *some eigenvalues are non-positive*, in which case the corresponding eigenvectors are included in the set of β_j and n is increased accordingly. If, on the other hand, (iii) *the set* (8) *admits no solution*, we have shown that no positive definite Q exists.

In both cases (i) and (iii) we have reached a definite conclusion; in case (ii) we have to search for a new set of x_is, with an enlarged set of vectors β_j. Note that the additional inequalities will not be satisfied by the set of x_is obtained in the previous iteration, which ensures that some progress is made in each iteration. To maximize the improvement made at each stage, and thus speed up the calculations, we replace (8) with:

$$\beta_j^T \left(\sum_{i=1}^{s} Q_i x_i \right) \beta_j \geqslant \varepsilon \geqslant 0, \tag{9}$$

and search for the solution α of (9) which maximizes ε. It is also convenient to introduce a scaling condition of the type $\Sigma |x_i| = 1$. This calculation can be done by a standard Linear Programming sub-routine, provided of course that the variables x_is are replaced by $2s$ non-negative variables or, more efficiently, using the Revised Simplex algorithm.

To find a positive definite Q our algorithm performs a series of iterations. Each iteration requires the solution of a Linear Programming problem—based on (9)—with an ever increasing number of inequalities, followed by a calculation of eigenvalues and eigenvectors of the matrix Q defined by the solution of the L.P. The algorithm converges when either all eigenvalues of Q are positive [case (i)], or when the L.P. has no feasible solution [case (iii)]. Otherwise [case (ii)] a new iteration is required.

When our algorithm is applied to the matrix Q obtained for the assembly of Fig. 5, at the start there are $n = m = 3$ inequalities plus the scaling condition, and the variables are α_1, α_2 and ε. Iterations one and two yield the optimal solutions $\alpha_1 = 0.5714$, $\alpha_2 = -0.4286$ ($\varepsilon = 0.2143$), and $\alpha_1 = 0.9547$, $\alpha_2 = -0.0453$ ($\varepsilon = 0.0227$), respectively. Both these solutions correspond to matrices Q with one negative eigenvalue, and yielding an additional inequality at each iteration. The third iteration, with $n = 5$, finds *no feasible solution*. This result is in agreement with the analysis in Section 3; but has been obtained by an algorithm which is easy to implement on a digital computer.

Finally, it should be noted that, as for many *cutting plane algorithms*, it is not possible to show that the above scheme will converge in a finite number of steps, although usually the performance of such algorithms is satisfactory (Luenberger, 1984).

5. BRIEF REVIEW OF PREVIOUS WORK. DISCUSSION

A rather unusual aspect of our work has been the rediscovery of an early paper by Kötter (1912) which presents the first analytical method to check whether a pin-jointed framework, which has both $s > 0$ and $m > 0$, is "rigid", by which Kötter meant that its mechanisms are first-order infinitesimal, "according to the general rules of the calculus of variations". Kötter studied a general three-dimensional pin-jointed assembly and, building on previous work by Mohr and Föppl, based his analysis on the function

$$\Phi = \frac{1}{2} \sum_{p=1}^{b} \frac{t_p}{l_p} [(x_q - x_r)^2 + (y_q - y_r)^2 + (z_q - z_r)^2 - l_p^2]. \tag{10}$$

Here l_p is the length of bar p, connecting joint $q \equiv (x_q, y_q, z_q)$ to joint $r \equiv (x_r, y_r, z_r)$, when the assembly is in its initial configuration; t_p is the axial force in the bar. For small configuration changes, (10) is proportional to the strain energy stored in the assembly. By differentiating Φ w.r.t. the nodal coordinates, a set of nodal equilibrium equations are obtained and, in absence of external loads, sets of self-equilibrating bar forces can be computed from them. Kötter (1912, Section 4) shows that "rigid" assemblies are those for which $\delta^2\Phi$ is either always positive, or always negative, and also that only inextensional displacements, that is m "variables" only, need to be considered when checking the sign of $\delta^2\Phi$. Kötter shows the calculations for a cube with its four space diagonals: a framework with $j = 8$ and $b = 16$, which has one state of self-stress and turns out to be "rigid", in spite of having three distinct infinitesimal inextensional mechanisms.

Kötter comments that his approach could be extended to assemblies with $s > 1$ by considering s functions Φ_i, each related to one particular state of self-stress, and such that $\delta^2\Phi_1 = \cdots = \delta^2\Phi_{s-1} = 0$. The assembly is "rigid" if $\delta^2\Phi_s$ is either always positive, or always negative. These remarks are relevant to the present study and, although the practical implementation of this scheme might prove rather difficult, there are clear similarities between Kötter's line of attack and the scheme which we have developed in Sections 2 and 4. Indeed, it might be possible to prove rigorously that our formulation in Section 2 is, in effect, a more general form of Kötter's stability criterion.

We have found only two references to Kötter's study in the published literature: Pollaczek-Geiringer (1927) and Levi-Civita and Amaldi (1930). Rather surprisingly, the latter authors chose to conduct a purely geometric investigation of the set of constraint equations —each corresponding to a bar— to be satisfied by all inextensional displacements. Their approach is easier to follow and more general than Kötter's but, for first-order infinitesimal mechanisms, it results in a quadratic function equivalent to $\delta^2\Phi$ but with $2j$ variables instead of just m. Being free from any static considerations, the scheme by Levi-Civita and Amaldi poses no extra difficulty if $s > 1$. More recently, Kuznetsov (1975a,b) has reduced the size of the quadratic form used by Levi-Civita and Amaldi (1930) to m variables only, after noting that an infinitesimal mechanism would be in a state of *stable equilibrium* if self-stressing forces were introduced in the bars. The resulting scheme is an up-to-date version of Kötter's algorithm, with its use in practice being restricted to assemblies with $s = 1$. Besseling (1979) and Tanaka and Hangai (1986) have followed an approach based on Linear Algebra, and hence related to the present study, to derive from a stability criterion a quadratic form in m variables. A comparison of Kötter's results with subsequent publications by other authors shows that little progress has been made over the past 75 years, in spite of several, intermittent attempts.

In this paper we have shown that, given an assembly with s (> 0) independent states of self-stress, m (> 0) independent mechanisms forming the matrix $\mathbf{D}$ and associated sets of product-force vectors $\mathbf{P}_1, \ldots, \mathbf{P}_s$, the mechanisms are first-order infinitesimal if and only if there exists a set of coefficients α_i for which the quadratic form

$$\beta^T \left(\sum_{i=1}^{s} \mathbf{P}_i^T \mathbf{D}\alpha_i \right) \beta$$

is positive definite.

We have also presented an automatic procedure to search for a set of suitable x,s, based on alternate Linear Programming phases and eigenvalue/eigenvector searches. This algorithm enables us to apply the proposed test to any assembly, but we have pointed out that for assemblies with $s = 1$ a non-iterative procedure can be used instead.

Our method has two obvious advantages over methods proposed previously. First, our scheme makes use of physically-based quantities, e.g. mechanisms, states of self-stress, etc. rather than a second-order analysis of constraint equations in the manner of Kuznetsov (1975a). These elements of the algorithm are directly calculable, and they correspond to physical quantities which afford greater insight. The scheme for computing mechanisms and states of self-stress, described in our previous paper, has been extended (Kwan and Pellegrino, 1989) to structural assemblies which include beams, connected in various ways, and cables which run over several small frictionless pulleys. In principle, the method described in this paper can be applied to such assemblies as well. Second, our scheme provides for assemblies with any number of statical indeterminacies to be analyzed, as in Levi-Civita and Amaldi (1930); however we require much smaller matrices for our analysis. A further advantage of our calculations is that we obtain, as a by-product, a set of bar tensions which would provide first-order stiffness against all inextensional modes, if we were to pre-stress the assembly. This information can be of considerable value in the design of pre-stressed mechanisms.

Finally, we should note that the use of our sophisticated general algorithm is hardly justified for the example shown in Fig. 5. It is quite obvious that the configuration shown is a rather special one, in which $s = 2$ and $m = 3$, of an assembly which has $s = 0$ and $m = 1$ in most configurations; Tarnai (personal communication) refers to such special configurations as "points of bifurcation of compatibility". Similarly, the ring assembly shown in Fig. 2 of our previous paper has $s = 2$ and $m = 2$ at its point of bifurcation of compatibility, i.e. when the top four bars lie in a vertical plane, although normally $s = 1$ and $m = 1$. We have not yet found a non-trivial example of a kinematically indeterminate assembly having $s = 2$, or greater, which exhibits a "finite" mechanism; and we should be interested to hear from any readers who know of non-trivial and possibly three-dimensional examples.

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